

A single trapped ion in a finite range trap

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Abstract

This paper presents a method to describe dynamics of an ion confined in a realistic finite range trap. We model this realistic potential with a solvable one and we obtain dynamical variables (raising and lowering operators) of this potential. We consider coherent interaction of this confined ion in a finite range trap and we show that its center-of-mass motion steady state is a special kind of nonlinear coherent states. Physical properties of this state and their dependence on the finite range of potential are studied.

Keywords: Nonclassical property, Finite range trap, Trapped ion,
Nonlinear coherent state

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1. Introduction

Single trapped ions represent elementary quantum systems that are approximately isolated from the environment [1]. In these systems both internal electronic states and external center-of-mass motional states (external states) can be coupled to and manipulated by light fields. This makes the trapped ion systems suited for quantum optical and quantum dynamical studies under well-controlled conditions. Motivated by the strong analogy between cavity quantum electrodynamics and the trapped ion system, various theoretical and experimental proposals have been made on how to create nonclassical and arbitrary states of motion of trapped ions. Preparation of the number

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states [2], coherent, quadrature squeezed number states and superposition of the number states were considered in this system experimentally [3] and theoretically [4]. Experimental preparation of the Schrödinger-cat state was considered [5]. Theoretical schemes for generation of arbitrary center-of-mass motional states of a trapped ion is described in [6]. Moreover, the possibility of generation of even and odd coherent states of the center-of-mass motion of a trapped ion is considered [7]. A new scheme for preparation of nonclassical motional states of trapped ions is investigated in [8]. Recently, Preparation of Dicke states in an ion chain is considered theoretically and experimentally [9]. In addition to the above-mentioned attempts, preparation of different family of nonlinear coherent states [10], is also studied theoretically [11, 12].

On the other hand, the trapped ion system has found some applications in quantum information and quantum computation [13]. For quantum information processing by trapped ion, preparation of some special states is important. Among these states, entangled states have found crucial importance. Preparation of the entangled states of trapped ion is considered recently [14]. For quantum computation applications, preparation of the two-dimensional cluster-state is considered [15]. Because of the some similarities between the trapped ion system and the Jaynes-Cummings model [16], the trapped ion system is used to realize different generalizations of the Jaynes-Cummings model which have found some applications in quantum information [17].

In all of the above-mentioned efforts on the trapped ion system, it is assumed that the ion is confined in a harmonic oscillator-shaped potential while the dimension of this potential extended to infinity. Hence, the range of the confining trap is infinity. However, in the realistic experimental setup, the dimension of trap is finite and the realistic trapping potential is not a harmonic oscillator potential but the truncated and modified one within the extension of the trap. In this paper, we assume that the confining potential for ion has finite range. We will model this confining potential with a solvable one. By using the concept of the f -deformed oscillator [10], we try to consider the trapped ion in confining potential with finite range as an f -deformed oscillator and in this context we obtain raising and lowering operators (dynamical variables) of this potential. The finite range effects of this model can be used in traps of the order of nano-scale, called nano Paul traps, that are attracted a great deal of attention recently [18]. It is worth to note that the confining model potential which is considered here is used for other confined physical systems, such as the Bose-Einstein condensate [19], and carriers in a quantum well [20]. The f -deformed oscillator approach

where we have considered here, has been used before for some other confined systems [21].

This paper presents a method to describe dynamics of an ion confined in a finite range trap. We will show that stationary state of the center-of-mass motion of the trapped ion is a special kind of nonlinear coherent states where its properties depend on the range of the confining potential. The outline of the paper is as follows. Section 2 deals with scheme for model potential and in this section we will obtain dynamical variables of this potential in the context of the f -deformed oscillator. In Sec. 3 we propose coherent interaction of an ion confined in a finite range potential and we consider its dynamics in the steady state. In this section we will obtain an eigenvalue equation for the state of the center-of-mass motion of the ion. In Sec. 4 we summarize definition of the nonlinear coherent states and we will show that the steady state of the ion motion can be considered as a nonlinear coherent state. Physical properties of this system are investigated in this section. Section 5 is devoted to the conclusion.

2. Algebraic approach for a particle in a finite range potential

To consider an ion in a finite range trap, we try to model the potential energy function of the realistic trap by an analytically solvable potential. For comparing new results with previous ones we are looking for a potential which reduces to the harmonic oscillator potential in a specific limit of its parameters. A potential which has this property is the modified Pöschl-Teller (MPT) potential [22]. The MPT potential has the following form

$$V(x) = D \tanh^2\left(\frac{x}{\delta}\right), \quad (1)$$

where D is the depth of the well, δ determines the range of the potential and x gives the relative distance from the equilibrium position. The well depth, D , can be defined as $D = \frac{1}{2}m\omega^2\delta^2$, with mass of the particle m and angular frequency ω of the harmonic oscillator, so that, in the limiting case $D \rightarrow \infty$ (or $\delta \rightarrow \infty$), but keeping the product $m\omega^2$ finite, the MPT potential energy reduces to the harmonic potential energy, $\lim_{D \rightarrow \infty} V(x) = \frac{1}{2}m\omega^2x^2$. Solving the Schrödinger equation, the energy eigenvalues for the MPT potential are obtained as [23]

$$E_n = D - \frac{\hbar^2\omega^2}{4D}(s-n)^2, \quad n = 0, 1, 2, \dots, [s] \quad (2)$$

in which $s = (\sqrt{1 + (\frac{4D}{\hbar\omega})^2} - 1)/2$, and $[s]$ stands for the closest integer to s that is smaller than s . The MPT oscillator quantum number n can not be larger than the maximum number of bound states $[s]$, because of the dissociation condition $s - n \geq 0$. Detailed description about this energy spectrum can be found in [24]. By introducing a dimensionless parameter $N = \frac{4D}{\hbar\omega} = \frac{2m\omega\delta^2}{\hbar}$, the bound energy spectrum in equation (2) can be rewritten as

$$E_n = \hbar\omega[-\frac{n^2}{N} + (\sqrt{1 + \frac{1}{N^2}} - \frac{1}{N})n + \frac{1}{2}(\sqrt{1 + \frac{1}{N^2}} - \frac{1}{N})]. \quad (3)$$

The relation (3) shows a nonlinear dependence on the quantum number n , so that, different energy levels are not equally spaced. As is evident, N is a dimensionless parameter and from now we refer this parameter as the depth of the trap. It is clear that, in the limiting case $D \rightarrow \infty$ (or $N \rightarrow \infty$), the energy spectrum for the quantum harmonic oscillator will be obtained, i.e., $E_n = \hbar\omega(n + \frac{1}{2})$. This means that for finite values of D (or finite values of δ), we have a deformed quantum oscillator, which its natural deformation from the quantum harmonic oscillator can be amplified by decreasing D or N . Thus, the well depth of this potential that identifies its range, is used to approximate the harmonic oscillator potential and it can also be considered as a controllable physical deformation parameter. It is interesting to note that the dimensionless parameter N can also be written as $N = \frac{\delta}{\Delta x}$. Here $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$, is the ground state wave function spread which for typical traps is of the order of nanometer (nm) [1]. δ , that determines the range of the potential would be of the same order of magnitude as the ion-electrode distance in a Paul trap system. It results that if trap size be of the order of nm , the finite range effects of the trap would be important. Such kind of the Paul traps have considered recently [18].

It is shown that [21], each quantum system which has an unequal spaced energy spectrum can be considered as an f -deformed oscillator. Therefore, according to the energy spectrum of the MPT potential, this system can be considered as an f -deformed oscillator [24]. On the other hand, the f -deformed quantum oscillator [10], as a nonlinear oscillator with a specific kind of nonlinearity, is characterized by the following deformed dynamical variables \hat{A} and \hat{A}^\dagger

$$\begin{aligned} \hat{A} &= \hat{a}f(\hat{n}) = f(\hat{n} + 1)\hat{a}, \\ \hat{A}^\dagger &= f(\hat{n})\hat{a}^\dagger = \hat{a}^\dagger f(\hat{n} + 1), \quad \hat{n} = \hat{a}^\dagger\hat{a}, \end{aligned} \quad (4)$$

where \hat{a} and \hat{a}^\dagger are usual boson annihilation and creation operators ($[\hat{a}, \hat{a}^\dagger] = 1$), respectively. The real deformation function $f(\hat{n})$ is a nonlinear operator-valued function of the harmonic number operator \hat{n} , which introduces some nonlinearities to the system. From equation (4), it follows that the f -deformed operators \hat{A} , \hat{A}^\dagger and \hat{n} satisfy the following closed algebra

$$\begin{aligned} [\hat{A}, \hat{A}^\dagger] &= (\hat{n} + 1)f^2(\hat{n} + 1) - \hat{n}f^2(\hat{n}), \\ [\hat{n}, \hat{A}] &= -\hat{A}, \quad [\hat{n}, \hat{A}^\dagger] = \hat{A}^\dagger. \end{aligned} \quad (5)$$

The above-mentioned algebra, represents a deformed Heisenberg-Weyl algebra whose nature depends on the nonlinear deformation function $f(\hat{n})$. An f -deformed oscillator is a nonlinear system characterized by a Hamiltonian of the harmonic oscillator form

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{A}^\dagger \hat{A} + \hat{A} \hat{A}^\dagger). \quad (6)$$

Using equation (4) and the number state representation $\hat{n}|n\rangle = n|n\rangle$, the eigenvalues of the Hamiltonian (6) can be written as

$$E_n = \frac{\hbar\omega}{2}[(n + 1)f^2(n + 1) + nf^2(n)]. \quad (7)$$

It is worth noting that in the limiting case $f(n) \rightarrow 1$, the deformed algebra (5) and the deformed energy eigenvalues (7) will reduce to the conventional Heisenberg-Weyl algebra and the harmonic oscillator spectrum, respectively.

Comparing the bound energy spectrum of the MPT oscillator, equation (3), and the energy spectrum of an f -deformed oscillator, equation (7), we obtain the corresponding deformation function for the MPT oscillator as

$$f^2(\hat{n}) = \sqrt{1 + \frac{1}{N^2} - \frac{\hat{n}}{N}}. \quad (8)$$

Furthermore, the ladder operators of the bound eigenstates of the MPT Hamiltonian can be written in terms of the conventional operators \hat{a} and \hat{a}^\dagger as follows

$$\hat{A} = \hat{a} \sqrt{\sqrt{1 + \frac{1}{N^2} - \frac{\hat{n}}{N}}}, \quad \hat{A}^\dagger = \sqrt{\sqrt{1 + \frac{1}{N^2} - \frac{\hat{n}}{N}}} \hat{a}^\dagger. \quad (9)$$

These two operators satisfy the deformed Heisenberg-Weyl commutation relation

$$[\hat{A}, \hat{A}^\dagger] = \sqrt{1 + \frac{1}{N^2}} - \frac{2\hat{n} + 1}{N}, \quad (10)$$

As is clear, in the limiting case $f(n) \rightarrow 1$ ($N \rightarrow \infty$) this deformed commutation relation will reduce to the conventional commutation relation, $[\hat{a}, \hat{a}^\dagger] = 1$.

As a result, in this section we conclude that the trapped ion in MPT potential can be considered as an f -deformed oscillator with specific kind of the f -deformed Heisenberg-Weyl algebra.

In the following, we will consider coherent interaction of a single trapped ion in a finite range trap with light fields. Then, we will generate the non-linear coherent states of ionic vibrational motion in a finite range trap and finally we will investigate some physical properties of these states such as, their number distribution, quadrature squeezing and their phase-space distribution.

3. Ion dynamics in a finite range trap

As is usual in theoretical consideration of trapped ion systems, the confining potential is assumed to be a spatial varying high-frequency time-dependent field, the so-called Paul trap, $V(\vec{r}, t)$. It is shown that, motion of a particle inside a such high-frequency trap can be treated by averaging over the fast motion (part of the particle displacement that its frequency is the same as frequency of trap fields). In this approach a confined particle in such a trap experiences a spatial static effective potential [25]. Usually this static potential is assumed to be a three dimensional harmonic oscillator-like potential so that in one direction (x -direction) can be written as $V(x) = \frac{1}{2}m\omega^2 x^2$ [1]. As is conventional, ion is cooled to the ground-state of the trap and in this situation due to smallness of the ratio of trap height to other energy scales, such as energy distance between two adjacent energy levels of the trap, the trap is assumed extend to infinity. However, in the realistic experimental setup, the dimension of the trap is finite and the realistic trapping potential is not a harmonic oscillator potential extending to infinity but the truncated and modified one within the extension of the trap. Thus, the realistic confining potential becomes flat near the edge of the trap and can be simulated by the tanh-shaped potential, so that in one dimension (x -direction) can be written as $V(x) = D \tanh^2(\frac{x}{\delta})$. In this paper, we try to investigate some effects which originate from finite range property of the trap.

According to the previous section, we model this trapped ion as an f -deformed quantum oscillator. Therefore, the oscillator-like Hamiltonian of this system can be written as

$$\hat{H}_t = \frac{\hbar\omega}{2}(\hat{A}\hat{A}^\dagger + \hat{A}^\dagger\hat{A}), \quad (11)$$

where we interpret the operator \hat{A} (\hat{A}^\dagger) as the operator whose action causes the transition of the ion center-of-mass motion to the lower (upper) energy state of the trap. These operators are given in Eq. (9). In fact, the Hamiltonian (11) is related to the external degrees of freedom of the ion. According to the resonant condition, the ion is assumed as a two-level system with the ground state $|g\rangle$ and the excited state $|e\rangle$. Then, internal degrees of freedom of the ion can be expressed with electronic flip operators $\hat{S}_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\hat{S}^+ = |e\rangle\langle g|$ and $\hat{S}^- = |g\rangle\langle e|$ which satisfy the usual $su(2)$ algebra. On the other hand, with the help of the suitable laser fields, the internal levels of the trapped ion can be coherently coupled to each other and to the external motional degrees of freedom of the ion. Therefore, the total Hamiltonian of the system may be given as

$$\hat{H} = \hat{H}_0 + \hat{H}_{int}(t), \quad (12)$$

where $\hat{H}_0 = \hat{H}_t + \hbar\omega_i\hat{S}_z$, with H_t given in Eq. (11), describes the free motion of the internal and external degrees of freedom of the ion. Here, $\hbar\omega_i$ refers to the energy difference of internal states of the ion, $\hbar\omega_i = E_e - E_g$. The interaction of the ion with the laser fields is described by $\hat{H}_{int}(t)$ and is written as

$$\hat{H}_{int}(t) = g [E_0 e^{-i(k_0\hat{x} - \omega_i t)} + E_1 e^{-i[k_1\hat{x} - (\omega_i - \omega_n)t]}] \hat{S}^+ + H.c., \quad (13)$$

in which g is coupling constant, k_0 and k_1 are the wave numbers of the driving laser fields and ω_n refers to the energy of the lower vibrational side-band with respect to the electronic transition of the ion. ω_n is the frequency of the ion transition between energy levels of the finite range trap. Because energy spectrum of the trap depends on the energy level numbers and we consider a transition between specific side-band levels, hence, we show the transition frequency with definite dependence to n . In the above Hamiltonian, $\hat{H}_{int}(t)$, \hat{x} is the operator of the center-of-mass position and may be defined as [21]

$$\hat{x} = \frac{\eta}{k_l}(\hat{A} + \hat{A}^\dagger), \quad (14)$$

where η being the Lamb-Dicke parameter and k_l is associated wave number to the characteristic length of the trap and assume to be $k_l \simeq k_0 \simeq k_1$. The interaction Hamiltonian (13) can be written as

$$\hat{H}_{int}(t) = \hbar e^{i\omega_i t} [\Omega_0 + \Omega_1 e^{-i\omega_n t}] e^{i\eta(\hat{A} + \hat{A}^\dagger)} \hat{S}^+ + H.c. , \quad (15)$$

$\Omega_0 = \frac{gE_0}{\hbar}$ and $\Omega_1 = \frac{gE_1}{\hbar}$ are the Rabi frequencies of the laser fields tuned to the electronic transition and the lower sideband, respectively. The interaction Hamiltonian in the interaction picture with respect to the \hat{H}_0 can be written as

$$\hat{H}_I = \hbar \Omega_1 \hat{S}^+ \left[\frac{\Omega_0}{\Omega_1} + e^{-i\omega_n t} \right] \exp \left[i\eta \left(e^{-i\hat{\nu}_n t} \hat{A} + \hat{A}^\dagger e^{i\hat{\nu}_n t} \right) \right] + H.c. , \quad (16)$$

where $\hat{\nu}_n = \frac{\omega}{2}[(\hat{n} + 2)f^2(\hat{n} + 2) - \hat{n}f^2(\hat{n})]$. In this relation the function $f(\hat{n})$ is given by Eq. (8).

By using the vibrational rotating-wave approximation [11] and applying the disentangling approach in [26] for the exponential term which appeared in equation (16), the interaction Hamiltonian (16) may be written as

$$\hat{H}_I = \hbar \Omega_1 \hat{S}^+ \left[\frac{\Omega_0}{\Omega_1} F_0(\hat{n}, \eta) + g(\eta) F_1(\hat{n}, \eta) \hat{a} \right] + H.c. , \quad (17)$$

where the function $F_j(\hat{n}, \eta)$ ($j = 0, 1$) is defined by

$$F_j(\hat{n}, \eta) = \sum_{l=0}^n \frac{[g(\eta)]^{2l}}{l!(l+j)!} \frac{f(\hat{n})! f(\hat{n}+j)!}{[f(\hat{n}-l)!]^2} \frac{\hat{n}!}{(\hat{n}-l)!} M(\hat{n}-l). \quad (18)$$

In this equation different functions are appeared which are defined as follows

$$\begin{aligned} g(\eta) &= \frac{i}{\sqrt{\gamma}} \tan(\sqrt{\gamma}\eta), & X_n &= \beta - \gamma(2n+1), \\ M(n) &= e^{-\frac{X_n}{\gamma} \ln(\cos(\sqrt{\gamma}\eta))}, \end{aligned} \quad (19)$$

where $\gamma = \frac{1}{N}$, $\beta = \sqrt{1 + \frac{1}{N^2}}$ and \hat{n} is an operator whose eigenvalues, n , refer to the excitation energy level number inside the trap. It is worth to note that in the limiting case $N \rightarrow \infty$ which is equivalent to $f(n) \rightarrow 1$, the system will reduce to the confined ion in the harmonic oscillator-shaped trap, which has been considered in [11]. The function $F_j(\hat{n}, \eta)$, given in Eq. (18), will reduce

to its counterpart in the harmonic oscillator-shaped trap [11].

The time evolution of the system is characterized by the master equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_I, \hat{\rho}] + \frac{\Gamma}{2}(2\hat{S}^-\hat{\rho}'\hat{S}^+ - \hat{S}^+\hat{S}^-\hat{\rho} - \hat{\rho}\hat{S}^+\hat{S}^-), \quad (20)$$

where Γ is the spontaneous emission rate. To account for the recoil of spontaneously emitted photons the first term of the damping part of the master equation contains

$$\hat{\rho}' = \frac{1}{2} \int_{-1}^1 dz Y(z) e^{ik_l \hat{x} z} \hat{\rho} e^{-ik_l \hat{x} z}, \quad (21)$$

$Y(z)$ is the angular distribution of the spontaneous emission and $\hat{\rho}$ is the vibronic density operator.

In the long-time limit, the ion will be populated in the ground state $|g\rangle$ as a consequence of atomic spontaneous emission. In this case, the steady-state solution of the master equation (20) can be assumed to be $\hat{\rho}_{ss} = |g\rangle|\psi\rangle\langle\psi|\langle g|$, where $|\psi\rangle$ stands for the vibronic motion of the ion. The stationary solution of Eq. (20) can be found by setting $\frac{d\hat{\rho}}{dt} = 0$ and since

$$\hat{S}^-|g\rangle\langle g| = \hat{S}^+\hat{S}^-|g\rangle\langle g| = |g\rangle\langle g|\hat{S}^+\hat{S}^- = 0, \quad (22)$$

we obtain

$$[\hat{H}_I, \hat{\rho}_{ss}] = 0. \quad (23)$$

From this equation, we find that the vibronic state $|\psi\rangle$ satisfies the following equation

$$\hat{a}h(\hat{n})|\psi\rangle = \chi|\psi\rangle, \quad \chi = -\frac{\Omega_0}{g(\eta)\Omega_1}. \quad (24)$$

In this equation $h(\hat{n}) = F_1(\hat{n} - 1, \eta)/F_0(\hat{n} - 1, \eta)$.

4. Nonlinear coherent states of ionic vibrational motion and their physical properties

Similar to the definition of the canonical coherent states [27], the coherent state of a generalized f -deformed oscillator is defined as a right-hand eigenstate of the generalized annihilation operator ($\hat{A} = \hat{a}f(\hat{n})$) as follows

$$\hat{A}|\alpha, f\rangle = \alpha|\alpha, f\rangle. \quad (25)$$

Due to the appearance of nonlinear deformation function, $f(\hat{n})$, in definition of these states, they are called nonlinear coherent states. According to this definition, vibronic state of the ion in the steady state, Eq. (24), is a nonlinear coherent state with

$$\begin{aligned} f(\hat{n}) &= h(\hat{n}) = \frac{F_1(\hat{n} - 1, \eta)}{F_0(\hat{n} - 1, \eta)}, \\ \alpha &= \chi = -\frac{\Omega_0}{g(\eta)\Omega_1}. \end{aligned} \quad (26)$$

Nonlinear coherent states can be expanded in terms of the usual Fock states ($\hat{n}|n\rangle = n|n\rangle$) as follows

$$|\alpha, f\rangle = N_f \sum_n \frac{\alpha^n}{\sqrt{n!f(n)!}} |n\rangle, \quad N_f = \left[\sum_n \frac{|\alpha|^{2n}}{n![f(n)!]^2} \right]^{-\frac{1}{2}}, \quad (27)$$

where $f(n)! = f(n)f(n-1)\cdots f(0)$. Thus, the steady state of the ion in Eq. (24) is a special kind of the nonlinear coherent state where its properties are defined by the function $h(\hat{n})$. This function is characterized by the Lamb-Dicke parameter η and quantum number n which refers to the level of vibronic excitation. Moreover, according to the Eq. (24), nonlinear coherent state of the ion depends on the complex parameter χ , which is controlled by the Rabi frequencies of the lasers, the Lamb-Dicke parameter and γ parameter that governed by the range of the trap. In order to get some insight about physical properties of this family of nonlinear coherent state, we consider some statistical properties of this state. In Fig. (??) we show the vibrational number distribution of this state, $p(n) = |\langle n|\psi\rangle|^2$. In all of the plots in this figure, Lamb-Dicke parameter and the ratio $\frac{\Omega_0}{\Omega_1}$ are chosen as $\eta = 0.22$ and $\frac{\Omega_0}{\Omega_1} = 0.85$, respectively. It can be seen that the vibrational number distribution depends sensitively on the depth (or range) of the trap. In some cases it is possible to prepare a superposition of several Fock states. Another feature of this figure is that by choosing the proper values of the depth of the trap, such as ($N = 30$), it is possible to prepare a superposition of two or three Fock states. An interesting property of this vibrational number distribution is that we can prepare a highly excited Fock state for external motion of the ion ($N = 45$) [28]. In this case with most probability we can claim that one Fock state is prepared. By increasing the

depth of the trap ($N = 75$), the vibrational number distribution will reduce to a superposition of Fock states again. In this case the distribution of the Fock states is approximately symmetric about the most probable number state. Thus, it is shown that for definite values of physical parameters, η and $\frac{\Omega_0}{\Omega_1}$ and for different values of the trap depth, we can prepare different states even a highly excited Fock state.

In Fig. (??) we have plotted quadrature squeezing of the state $|\psi\rangle$, Eq. (24). Physical parameters for this plot are chosen as $\eta = 0.25$, $\frac{\Omega_0}{\Omega_1} = 0.31$ and the phase of the quadrature operator is chosen as $\frac{\pi}{4}$. This figure depicts squeezing behavior versus the depth of the trap. It is evident that for some values of the depth, the state (24) exhibits quadrature squeezing. Hence, in addition to the remarkable properties of the vibrational number distribution, this state has other nonclassical property. The non-classical properties of nonlinear coherent states is one of their most important properties [29].

In Fig. (??), we have shown the contour plots of the Q function of the state (24). In this figure, different plots belong to different depths of the trap with $\eta = 0.75$ and $\frac{\Omega_0}{\Omega_1} = 0.9$. In the case of $N = 7$ (plot (a)) the plot contains contribution at several amplitudes. This feature implies occurrence of quantum interference effects inherent in this state. It displays several localized regions where it becomes extremely small. This phenomena is related to the separate peaks of the number distribution of state (24) which are rather close together. By increasing the depth of the trap, in plot (b), $N = 26$, and plot (c), $N = 45$, this strong structure of the Q function is disappeared. In these cases the Q function has one peak and this shows that the peaks of the number distribution are decreased. On the other hand, the cross section of the Q function is not symmetric and this shows that for selected values of the parameters, the associated quadrature operator exhibits quadrature squeezing. With more increasing the depth of the trap, in plot (d), $N = 75$, structure of the Q function becomes stronger than plots (b) and (c). In this case, the state exhibits quadrature squeezing and we expect that quantum interference occurs again.

To obtain more information about the nature of the state (24), we have considered its associated Wigner function, $W(\alpha)$. The Wigner function for different values of the Lamb-Dicke parameter and the depth of the trap is shown in Fig. (??). In this figure the ratio $\frac{\Omega_0}{\Omega_1}$ is chosen equal to 0.9. The negative values of the Wigner function are a signature of the nonclassical nature of the associated state. As is seen, in all cases the Wigner function has negative values. To consider the Lamb-Dicke parameter effects, in plots

??(a)-??(c), we have decreased the Lamb-Dicke parameter while the depth of the trap is chosen constant. The Wigner function in plot ??(a) shows occurrence of the quantum interference. Decreasing of the Lamb-Dicke parameter splits peaks of the Wigner function in two groups. This yields a coherent superposition of two quantum states. It is evident that decreasing of the Lamb-Dicke parameter will decrease the amplitude of the Wigner function. In addition to the Lamb-Dicke parameter effects, dependence of the Wigner function to the depth of the trap is considered in plots ??(d)-??(f). It is seen that in plot ??(d), for selected parameters, the Wigner function is split into two parts which is signature of superposition of two coherent states, because each part consists of several peaks. By increasing the depth of the trap, these two parts are going to be mixed and the quantum interference will be occurred.

5. Conclusion

We have studied dynamics of a single trapped ion in a finite range trap. In the context of the f -deformed oscillators, we have shown that the confined ion in a finite range trap can be assumed as an f -deformed oscillator. By modelling the realistic potential with the modified Pöschl-Teller potential, we have obtained dynamical variables (raising and lowering operators) of this system. Moreover, we have proposed a scheme for preparation of a special family of nonlinear coherent states. Such states could be generated as stationary states of the center-of-mass motion of a laser-driven trapped ion in a finite range trap while interacts with a bichromatic laser field. When the motional state is nonlinear coherent state, the ion is decoupled from the driving laser field. Then, any perturbation of this motional state leads to the switching of the interaction and this leads to a self-stabilization of the state. We have shown that the prepared motional state of the ion has some nonclassical features which strongly depend on the depth of the trap. These states show some coherence effects such as localization of their phase-space distribution and splitting to two or more sub-states which the latter leads to quantum interference. According to the profile of the Q function of these states, they exhibit quadrature squeezing and for specific values of the physical parameters we have calculated their quadrature squeezing. It is shown that the nonclassical nature of the prepared states depends on the depth of the trap so that for specific values of the depth, both quantum interference and quadrature squeezing will occur but for some other values,

this state exhibits quadrature squeezing only.

In view of interesting properties of generated states in this paper, states of this type and physical system under consideration might to be of more general interest. First of all, the single trapped ion in finite range trap has a finite dimensional Hilbert space. As mentioned before, the number of energy levels in this system is controlled by the depth of the trap. As we know, size of the Hilbert space (dimension of the Hilbert space) has a crucial importance in some quantum phenomena, such as decoherence. Due to the development in experimental set ups of trapped ion, it seems possible to organize an experiment to consider Hilbert space size effects for this system. Then, our system can be considered as an experimental set up to investigate Hilbert space size effects. Second, this system turn out to be of interest for realization of the quantum groups. If we take a look at Hamiltonian (13), it seems that in the Lamb-Dicke regime ($\eta \ll 1$), this system can be considered as a realization of a deformed Jaynes-Cummings model. By considering the Lamb-Dicke limit, the exponential in Eq. (13) can be expanded to lowest order, resulting in the operator $g'(\hat{A}\hat{S}^+ + \hat{A}^\dagger\hat{S}^-)$, which corresponds to the deformed Jaynes-Cummings model (in this relation $g' = \eta g$). In addition, it is shown that there is a relation between the operators \hat{A} and \hat{A}^\dagger in Eq. (9) and the q -deformed algebra [24]. Therefore, our model can be considered as a realization of q -deformed and general deformed Jaynes-Cummings model where Lamb-Dicke parameter plays an important role on this issue. Third, in recent types of the Paul traps, the so-called nano Paul traps [18], the finite range effects of trapping potential are more important. It seems that our model which tries to consider finite range effects can provide a theoretical description for investigating the nano Paul traps. To put every things in a nut shell, our model in this paper provides an experimental set up to consider Hilbert space size effects and realization of q -deformed and general f -deformed algebras.

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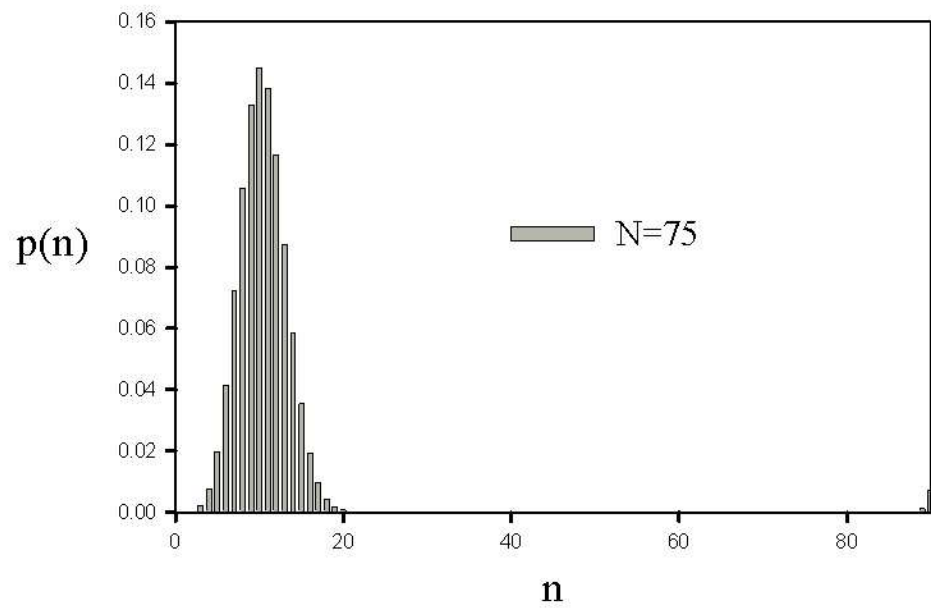
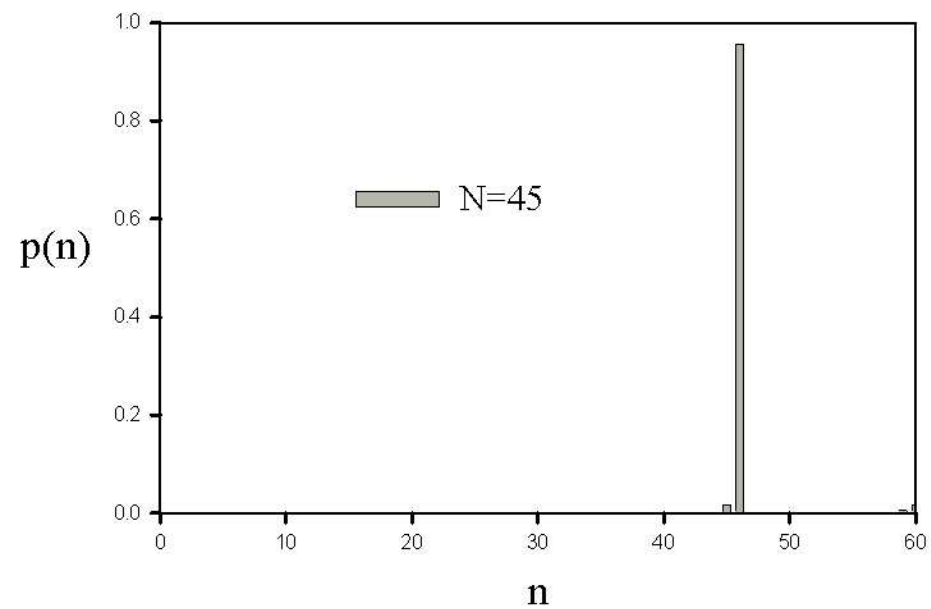
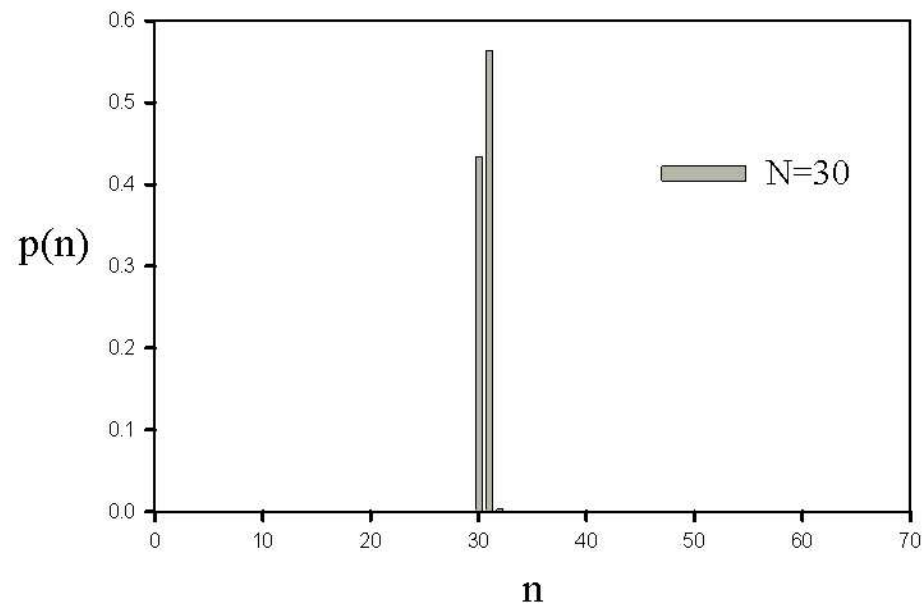
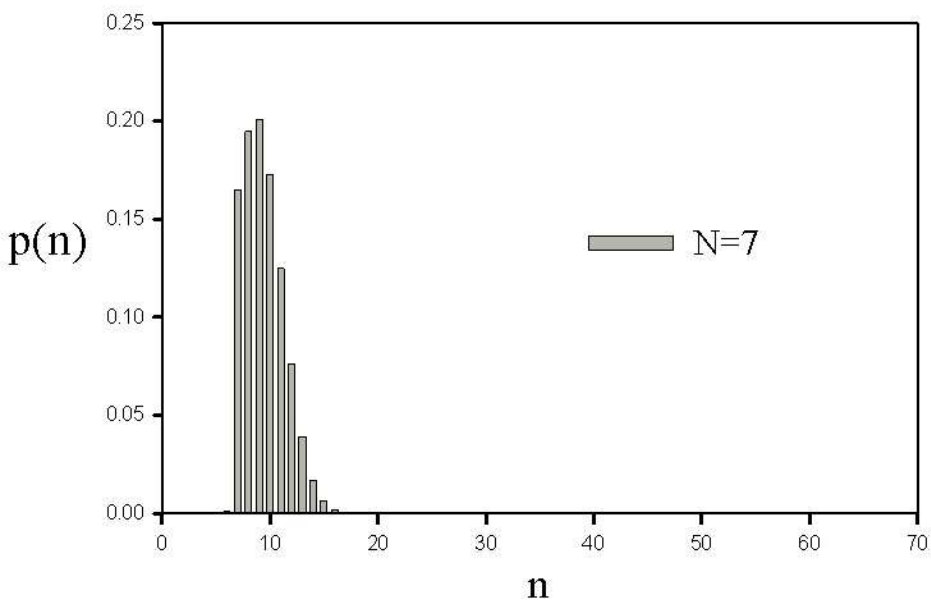
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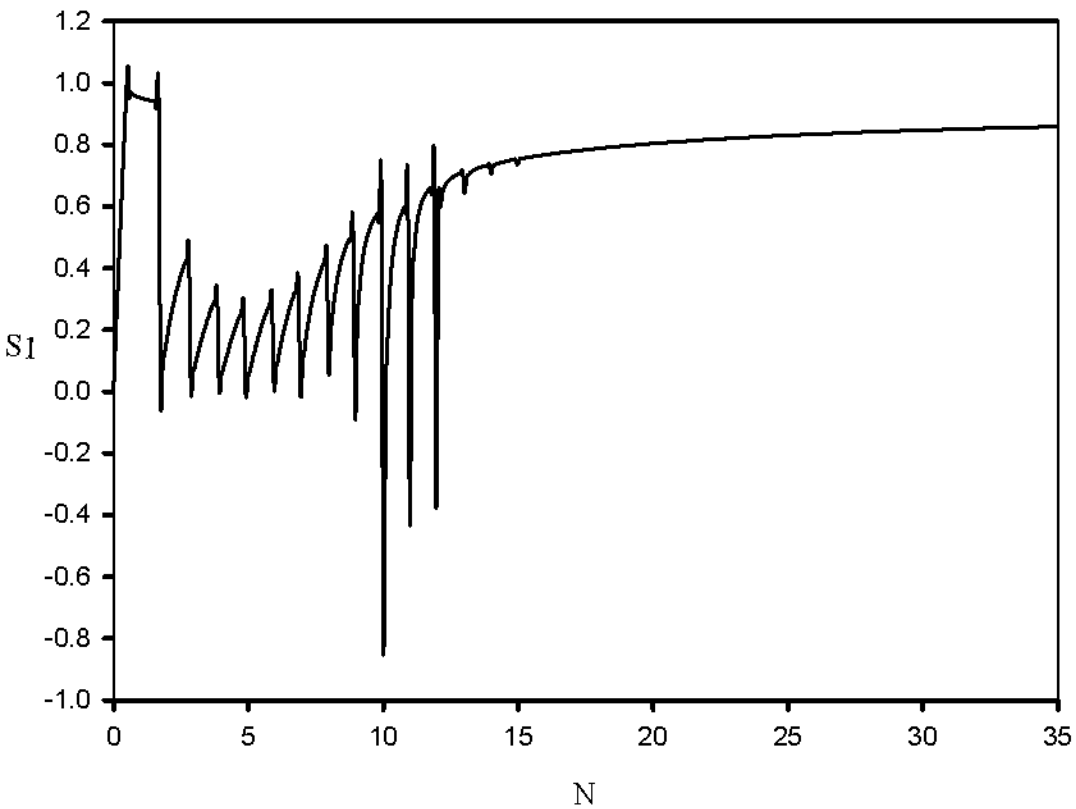
Fig. 1. The vibrational number distribution is shown for four values of the depth of the trap. The values of the depth, N , are written on each plot and $\eta = 0.22$ and $\frac{\Omega_0}{\Omega_1} = 0.85$.

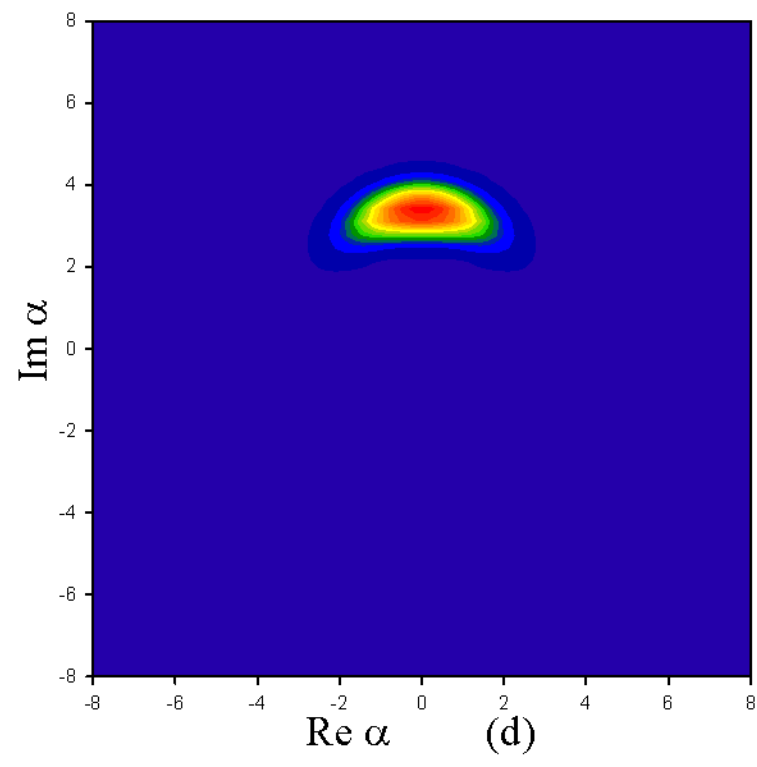
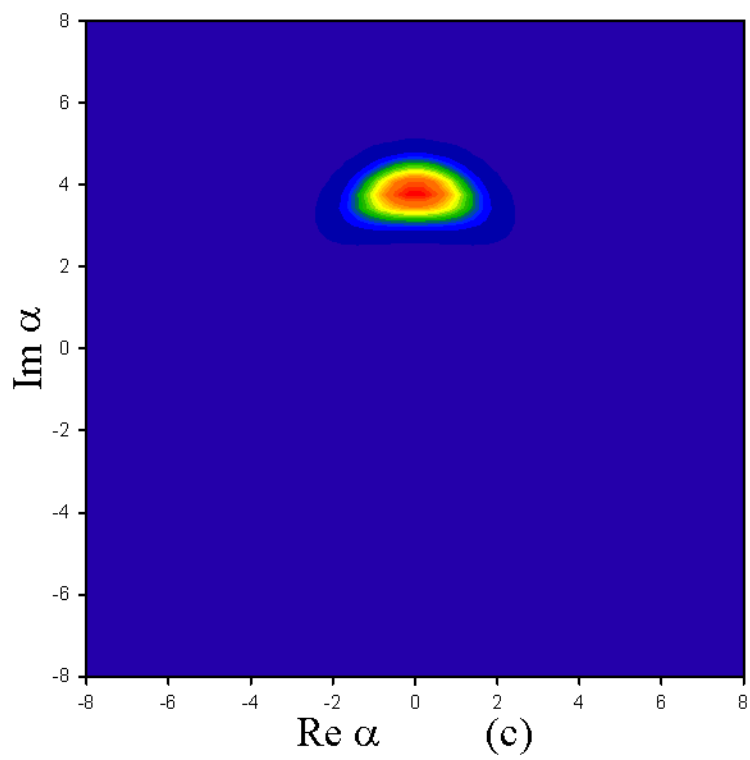
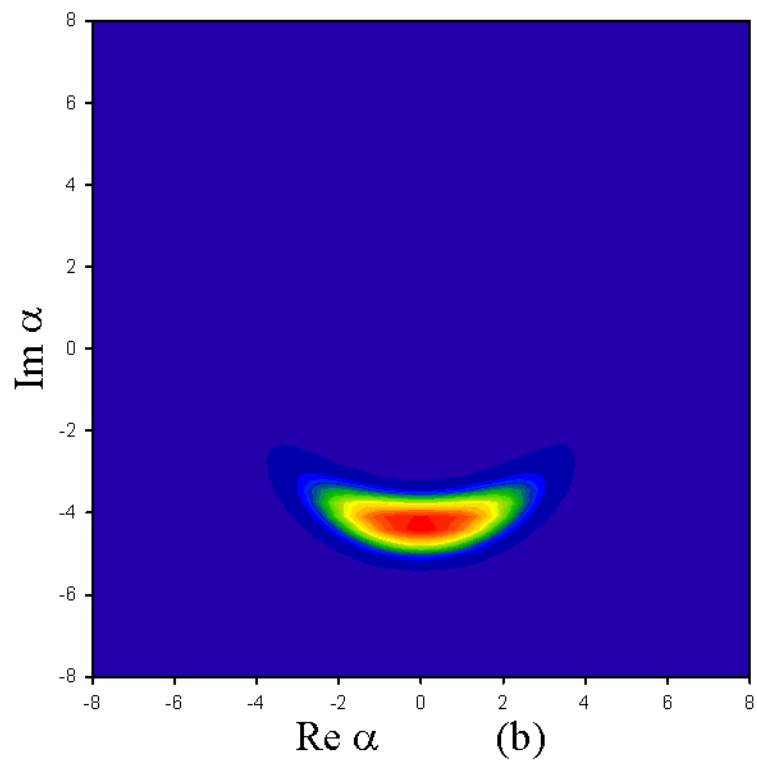
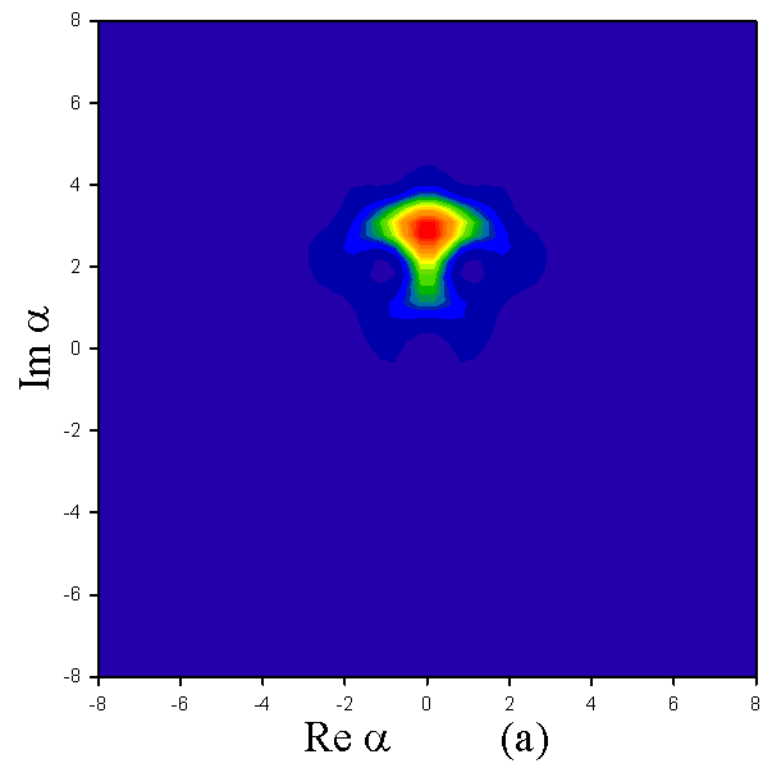
Fig. 2. Plot of quadrature squeezing versus depth of the trap. In this plot $\eta = 0.25$, $\frac{\Omega_0}{\Omega_1} = 0.31$ and quadrature operator phase is selected as $\frac{\pi}{4}$.

Fig. 3. Contour plots of the Q function for $\eta = 0.75$ and $\frac{\Omega_0}{\Omega_1} = 0.9$. In this figure light region indicates large values of the function. Each plot belongs to specific values of the depth of the trap. In plot (a) $N = 7$, plot (b) $N = 26$, plot (c) $N = 45$ and in plot (d) the depth of the trap is selected as $N = 75$.

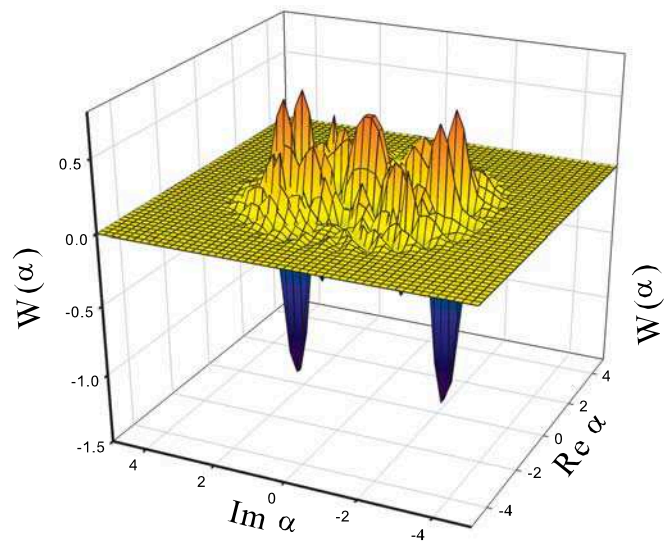
Fig. 4. Plots of the Wigner function for different values of the Lamb-Dicke parameter and the depth of the trap which are shown on each plot. In all plots the ratio $\frac{\Omega_0}{\Omega_1}$ is selected equal to 0.9.



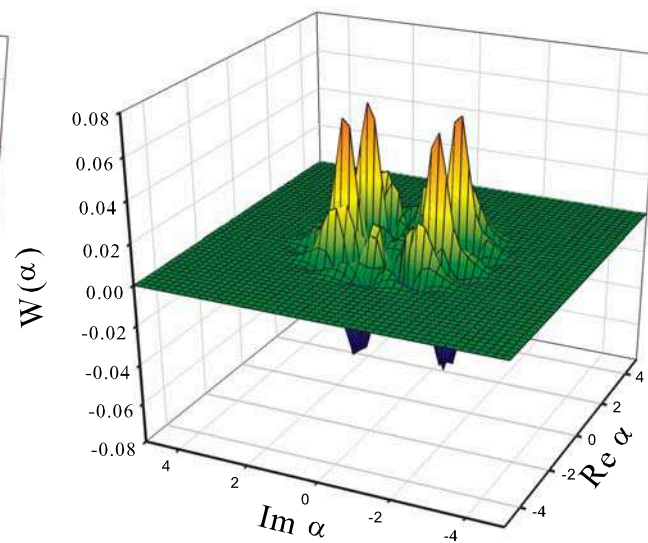




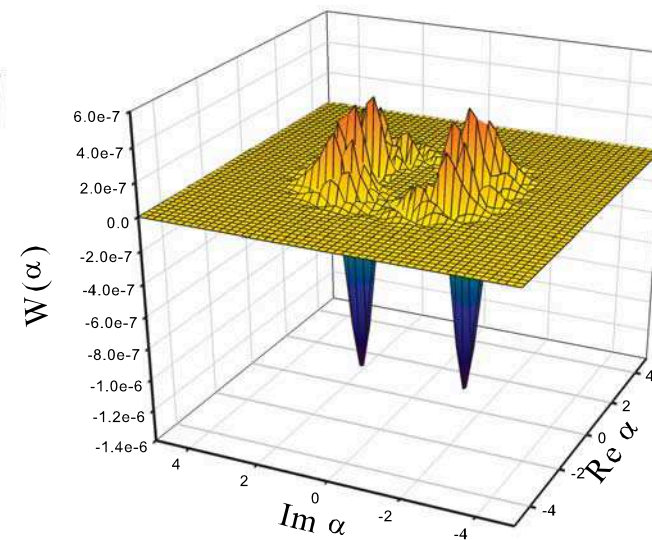
a) $N=7, \eta=0.75$



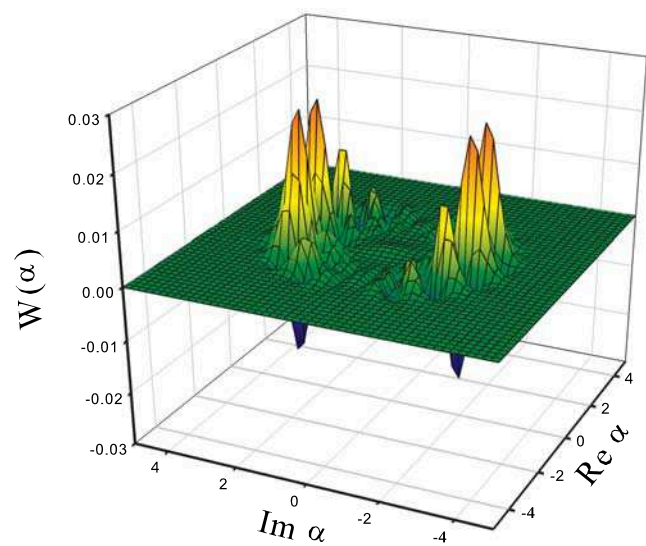
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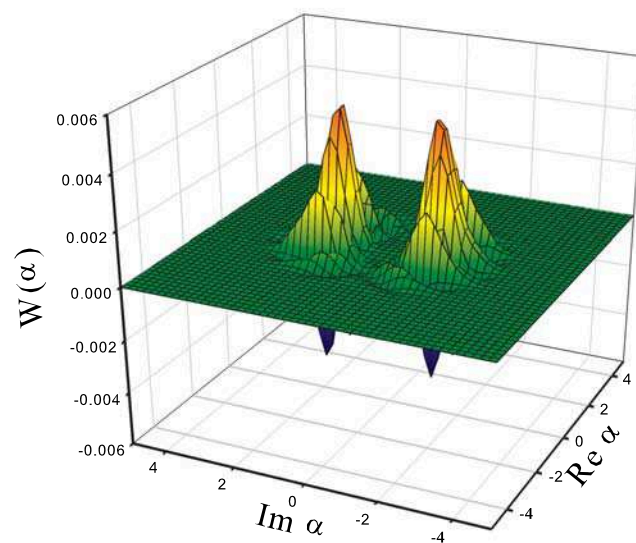
c) $N=7, \eta=0.55$



d) $N=25, \eta=0.75$



e) $N=45, \eta=0.75$



f) $N=75, \eta=0.75$

